Tight adversary bounds for composite functions

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Abstract

The quantum adversary method is a versatile method for proving lower bounds on quantum algorithms. It yields tight bounds for many computational problems, is robust in having many equivalent formulations, and has natural connections to classical lower bounds. A further nice property of the adversary method is that it behaves very well with respect to composition of functions. We generalize the adversary method to include costs—each bit of the input can be given an arbitrary positive cost representing the difficulty of querying that bit. We use this generalization to exactly capture the adversary bound of a composite function in terms of the adversary bounds of its component functions. Our results generalize and unify previously known composition properties of adversary methods, and yield as a simple corollary the $\Omega(\sqrt{n})$ bound of Barnum and Saks on the quantum query complexity of read-once functions.

1 Introduction

One of the most successful methods for proving lower bounds on quantum query complexity is via adversary arguments. The basic idea behind the adversary method is that if a query algorithm successfully computes a Boolean function f, then in particular it is able to "distinguish" 0-inputs from 1-inputs. There are many different ways to formulate the progress an algorithm makes in distinguishing 0-inputs from 1-inputs by making queries — these varying formulations have led to several versions of the adversary method including Ambainis' original weight schemes [Amb02, Amb03], the Kolmogorov complexity method of Laplante and Magniez [LM04], and a bound in terms of the matrix spectral norm due to Barnum, Saks and Szegedy [BSS03]. Using the duality theory of semidefinite programming, Špalek and Szegedy [ŠS06] show that in fact all of these formulations are equivalent.

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We will primarily use the spectral formulation of the adversary method. Let $Q_2(f)$ denote the two-sided bounded-error query complexity of a Boolean function $f: S \to \{0,1\}$, with $S \subseteq \{0,1\}^n$. Let Γ be a symmetric matrix with rows and columns labeled by elements of S. We say that Γ is an adversary matrix for f if $\Gamma[x,y]=0$ whenever f(x)=f(y). The spectral adversary method states that $Q_2(f)$ is lower bounded by a quantity ADV(f) defined in terms of Γ .

Theorem 1 ([BSS03]) For any function $f: S \to \{0,1\}$, with $S \subseteq \{0,1\}^n$ and any adversary matrix Γ for f, let

$$\mathrm{ADV}(f) = \max_{\substack{\Gamma \geq 0 \\ \Gamma \neq 0}} \ \frac{\|\Gamma\|}{\max_i \|\Gamma \circ D_i\|}.$$

Then $Q_2(f) = \Omega(ADV(f))$.

Here D_i is the zero-one valued matrix defined by $D_i[x,y]=1$ if and only if bitstrings x and y differ in the i-th coordinate, and ||M|| denotes the spectral norm of the matrix M.

One nice property of the adversary method is that it behaves very well for iterated functions. For a function $f:\{0,1\}^n \to \{0,1\}$ we define the d-th iteration of f recursively as $f^1=f$ and $f^{d+1}=f\circ (f^d,\ldots,f^d)$ for $d\geq 1$. Ambainis [Amb03] shows that if $\mathrm{ADV}(f)\geq a$ then $\mathrm{ADV}(f^d)\geq a^d$. Thus by proving a good adversary bound on the base function f, one can easily obtain good lower bounds on the iterates of f. In this way, Ambainis shows a super-linear gap between the bound given by the polynomial degree of a function and the adversary method, thus separating polynomial degree and quantum query complexity.

Laplante, Lee, and Szegedy [LLS06] show a matching upper bound for iterated functions, namely that if $ADV(f) \leq a$ then $ADV(f^d) \leq a^d$. Thus we conclude that the adversary method possesses the following composition property.

Theorem 2 ([Amb03, LLS06]) For any function $f: S \to \{0, 1\}$, with $S \subseteq \{0, 1\}^n$ and natural number d > 0,

$$ADV(f^d) = ADV(f)^d$$
.

A natural possible generalization of Theorem 2 is to consider composed functions that can be written in the form

$$h = f \circ (g_1, \dots, g_k). \tag{1}$$

One may think of h as a two-level decision tree with the top node being labeled by a function $f:\{0,1\}^k \to \{0,1\}$, and each of the k internal nodes at the bottom level being labelled by a function $g_i:\{0,1\}^{n_i} \to \{0,1\}$. We do not require that the inputs to the inner functions g_i have the same length. An input $x \in \{0,1\}^n$ to h is a bit string of length $n = \sum_i n_i$, which we think of as being comprised of k parts, $x = (x^1, x^2, \dots, x^k)$, where $x^i \in \{0,1\}^{n_i}$. We may evaluate h on input x by first computing the x bits $x_i = g_i(x^i)$, and then evaluating x on input $x = (x_1, x_2, \dots, x_k)$.

It is plausible, and not too difficult to prove, that if $a_1 \leq \mathrm{ADV}(f) \leq a_2$ and $b_1 \leq \mathrm{ADV}(g_i) \leq b_2$ for all i, then $a_1b_1 \leq \mathrm{ADV}(h) \leq a_2b_2$. In particular, if the adversary bounds of all of the subfunctions g_i are equal (i.e., $\mathrm{ADV}(g_i) = \mathrm{ADV}(g_j)$ for all i, j), then we can give an exact expression for the adversary bound on h in terms of the adversary bounds of its sub-functions,

$$ADV(h) = ADV(f) \cdot ADV(g_i),$$
 (2)

It is not so clear, however, what the exact adversary bound of h should be when the adversary bounds of the sub-functions g_i differ. The purpose of this paper is to give such an expression.

To do so, we develop as an intermediate step a new generalization of the adversary method by allowing input bits to be given an arbitrary positive cost. For any function $f: \{0,1\}^n \to \{0,1\}$, and any vector $\alpha \in \mathbb{R}^n_+$ of length n of positive reals, we define a quantity $ADV_{\alpha}(f)$ as follows:

$$ADV_{\alpha}(f) = \max_{\substack{\Gamma \geq 0 \\ \Gamma \neq 0}} \min_{i} \left\{ \alpha_{i} \frac{\|\Gamma\|}{\|\Gamma \circ D_{i}\|} \right\}.$$

One may think of α_i as expressing the cost of querying the i-th input bit x_i . For example, x_i could be equal to the parity of $2\alpha_i$ new input bits, or, alternatively, each query to x_i could reveal only a fraction of $1/\alpha_i$ bits of information about x_i . When $\alpha=(a,\ldots,a)$ and all costs are equal to a, the new adversary bound $\mathrm{ADV}_{\alpha}(f)$ reduces to $a \cdot \mathrm{ADV}(f)$, the product of a and the standard adversary bound $\mathrm{ADV}(f)$. In particular, when all costs a=1, we have the original adversary bound, and so $Q_2(f)=\Omega(\mathrm{ADV}_{\vec{1}}(f))$. When α is not the all one vector, then $\mathrm{ADV}_{\alpha}(f)$ will not necessarily be a lower bound on the quantum query complexity of f, but this quantity can still be very useful in computing the adversary bound of composed functions, as can be seen in our main theorem:

Theorem 3 (Exact expression for the adversary bound of composed functions) For any function $h: S \to \{0,1\}$ of the form $h = f \circ (g_1, \ldots, g_k)$ with domain $S \subseteq \{0,1\}^n$, and any cost function $\alpha \in \mathbb{R}^n_+$,

$$\mathrm{ADV}_{\alpha}(h) = \mathrm{ADV}_{\beta}(f),$$
 where $\beta_i = \mathrm{ADV}_{\alpha^i}(g_i)$, $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^k)$, and $\beta = (\beta_1, \dots, \beta_k)$.

The usefulness of this theorem is that it allows one to divide and conquer — it reduces the computation of the adversary bound for h into the disjoint subproblems of first computing the adversary bound for each g_i , and then, having determined $\beta_i = \text{ADV}(g_i)$, computing $\text{ADV}_{\beta}(f)$, the adversary bound for f with costs β .

One need not compute exactly the adversary bound for each g_i to apply the theorem. Indeed, a bound of the form $a \leq ADV(g_i) \leq b$ for all i already gives some information about h.

Corollary 4 If
$$h = f \circ (g_1, \dots, g_k)$$
 and $a \leq ADV(g_i) \leq b$ for all i , then $a \cdot ADV(f) \leq ADV(h) \leq b \cdot ADV(f)$.

One limitation of our theorem is that we require the sub-functions g_i to act on disjoint subsets of the input bits. Thus one cannot use this theorem to compute the adversary bound of any function by, say, proceeding inductively on the structure of a $\{\land,\lor,\neg\}$ -formula for the function. One general situation where the theorem can be applied, however, is to read-once functions, as by definition these functions are described by a formula over $\{\land,\lor,\neg\}$ where each variable appears only once. To demonstrate how Theorem 3 can be applied, we give a simple proof of the $\Omega(\sqrt{n})$ lower bound due to Barnum and Saks [BS04] on the bounded-error quantum query complexity of read-once functions.

Corollary 5 (Barnum-Saks) Let h be a read-once Boolean function with n variables. Then $Q_2(f) = \Omega(\sqrt{n})$.

Proof. We prove by induction on the number of variables n that $ADV(f) \ge \sqrt{n}$. If n = 1 then the formula is either x_i or $\neg x_i$ and taking $\Gamma = 1$ shows the adversary bound is at least 1.

Now assume the induction hypothesis holds for read-once formulas on n variables, and let h be given by a read-once formula with n+1 variables. As usual, we can push any NOT gates down to the leaves, and assume that the root gate in the formula for h is labeled either by an AND gate or an OR gate. Assume it is AND—the other case follows similarly. In this case, h can be written as $h=g_1 \wedge g_2$ where g_1 is a read-once function on $n_1 \leq n$ bits and g_2 is a read-once function on $n_2 \leq n$ bits, where $n_1+n_2=n+1$. We want to calculate $\mathrm{ADV}_{\vec{1}}(h)$. Applying Theorem 3, we proceed to first calculate $\beta_1=\mathrm{ADV}(g_1)$ and $\beta_2=\mathrm{ADV}(g_2)$. By the induction hypothesis, we know $\beta_1 \geq \sqrt{n_1}$ and $\beta_2 \geq \sqrt{n_2}$. We now proceed to calculate $\mathrm{ADV}_{\vec{1}}(h)=\mathrm{ADV}_{(\beta_1,\beta_2)}(\mathrm{AND})$. We set up our AND adversary matrix as follows:

	00	01	10	11
00	0	0	0	0
00 01 10	0	0	0	β_1
10	0	0	0	β_2
11	0	β_1	β_2	0

This matrix has spectral norm $\sqrt{\beta_1^2 + \beta_2^2}$, and $\|\Gamma \circ D_1\| = \beta_1$, and $\|\Gamma \circ D_2\| = \beta_2$. Thus

$$\beta_1 \frac{\|\Gamma\|}{\|\Gamma \circ D_1\|} = \beta_2 \frac{\|\Gamma\|}{\|\Gamma \circ D_2\|} = \sqrt{\beta_1^2 + \beta_2^2} \ge \sqrt{n+1}.$$

We prove Theorem 3 in two parts. Our main technical lemma is given in Section 2, where we show a general result about the behavior of the spectral norm under composition of adversary matrices; we use this lemma in Section 3 to show the lower bound $\mathrm{ADV}_{\alpha}(h) \geq \mathrm{ADV}_{\beta}(f)$. This lower bound is the only direction which is needed in Corollary 5, thus a self-contained proof of this result can be obtained by reading Section 2 and Section 3. In Section 4 we prove the upper bound $\mathrm{ADV}_{\alpha}(h) \leq \mathrm{ADV}_{\beta}(f)$. This is done by dualizing the spectral norm expression for ADV_{α} and showing how the dual solutions compose.

2 Spectral norm of a composition matrix

In this section we prove our main technical lemma. Given an adversary matrix Γ_f realizing the adversary bound for f and adversary matrices Γ_{g_i} realizing the adversary bound for g_i where $i=1,\ldots,k$, we build an adversary matrix Γ_h for the function $h=f\circ (g_1,\ldots,g_k)$. The main lemma expresses the spectral norm of this Γ_h in terms of the spectral norms of Γ_f and Γ_{g_i} .

Let Γ_f be an adversary matrix for f, i.e. a matrix satisfying $\Gamma_f[x,y]=0$ if f(x)=f(y), and let δ_f be a principal eigenvector of Γ_f with unit norm. Similarly, let Γ_{g_i} be a spectral matrix for g_i and let δ_{g_i} be a principal eigenvector of unit norm, for every $i=1,\ldots,k$.

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It is helpful to visualize an adversary matrix in the following way. Let $X_f = f^{-1}(0)$ and $Y_f = f^{-1}(1)$. We order the the rows first by elements from X_f and then by elements of Y_f . In this way, the matrix has the following form:

$$\Gamma_f = \left[\begin{array}{cc} 0 & \Gamma_f^{(0,1)} \\ \Gamma_f^{(1,0)} & 0 \end{array} \right]$$

where $\Gamma_f^{(0,1)}$ is the submatrix of Γ_f with rows labeled from X_f and columns labeled from Y_f and $\Gamma_f^{(1,0)}$ is the conjugate transpose of $\Gamma_f^{(0,1)}$.

Thus one can see that an adversary matrix for a Boolean function corresponds to a (weighted) bipartite graph where the two color classes are the domains where the function takes the values 0 and 1. For $b \in \{0,1\}$ let $\delta_{g_i}^{\dagger b}[x] = \delta_{g_i}[x]$ if $g_i(x) = b$ and $\delta_{g_i}^{\dagger b}[x] = 0$ otherwise. In other words, $\delta_{g_i}^{\dagger b}$ is the vector δ_{g_i} restricted to the color class b.

Before we define our composition matrix, we need one more piece of notation. Let $\Gamma_f^{(0,0)} = \|\Gamma_f\|I_{|X_f|}$, where I is a $|X_f|$ -by- $|X_f|$ identity matrix and $\Gamma_f^{(1,1)} = \|\Gamma_f\|I_{|Y_f|}$.

We are now ready to define the matrix Γ_h :

Definition 1
$$\Gamma_h[x,y] = \Gamma_f[\tilde{x},\tilde{y}] \cdot \left(\bigotimes_i \Gamma_{g_i}^{(\tilde{x}_i,\tilde{y}_i)}\right)[x,y]$$

A similar construction of Γ_h is used by Ambainis to establish the composition theorem for iterated functions.

Before going into the proof, we look at a simple estimate of the spectral norm of Γ_h . Notice that for any values $b_0, b_1 \in \{0, 1\}$ the matrix $\Gamma_{g_i}^{(b_0, b_1)}$ is a submatrix of $\Gamma_{g_i} + \|\Gamma_{g_i}\|I$. Thus the matrix Γ_h is a submatrix of the matrix

$$\Gamma_f \otimes \left(igotimes_f (\Gamma_{g_i} + \|\Gamma_{g_i}\|I) \right).$$

Therefore the spectral norm of Γ_h is upper bounded by the spectral norm of this tensor product matrix. Since $\|\Gamma_{g_i} + \|\Gamma_{g_i}\| I\| = 2\|\Gamma_{g_i}\|$ it follows that $\|\Gamma_h\| \leq \|\Gamma_f\| \cdot 2^k \prod_{i=1}^k \|\Gamma_{g_i}\|$.

By exploiting the block structure of Γ_h and the fact that Γ_h is nonnegative, we are able to prove the following tight bound, the key to our adversary composition theorem.

Lemma 6 Let Γ_h be defined as above for a nonnegative adversary matrix Γ_f . Then $\|\Gamma_h\| = \|\Gamma_f\| \cdot \prod_{i=1}^k \|\Gamma_{g_i}\|$ and a principal eigenvector of Γ_h is $\delta_h[x] = \delta_f[\tilde{x}] \cdot \prod_{i=1}^k \delta_{g_i}^{\upharpoonright \tilde{x}_i}[x]$.

Proof. First we will show $\|\Gamma_h\| \leq \|\Gamma_f\| \cdot \prod_{i=1}^k \|\Gamma_{g_i}\|$ by giving an upper bound on $u^*\Gamma_h u$ for an arbitrary unit vector u.

For $a \in \{0,1\}^k$ let $X_a = \{x \in \{0,1\}^n : \tilde{x} = a\}$. The 2^k many (possibly empty) sets X_a partition X. Let u_a be the vector u restricted to X_a , that is $u_a[x] = u[x]$ if $x \in X_a$ and $u_a[x] = 0$ otherwise. The sets $\{X_a\}_{a \in \{0,1\}^k}$ give rise to a partition of the matrix Γ_h into 2^{2k} many blocks, where block (a,b) is labelled by rows from X_a and columns from X_b . The (a,b) block of Γ_h is equal to the matrix $\Gamma_f[a,b] \cdot \otimes_{i=1}^k \Gamma_{g_i}^{(a_i,b_i)}$.

Now we have

$$u^*\Gamma_h u = \sum_{a,b} \Gamma_f[a,b] \cdot \sum_{\substack{x,y \\ x \in X_a, y \in X_h}} \left(\bigotimes_{i=1}^k \Gamma_{g_i}^{(a_i,b_i)} \right) [x,y] \cdot u[x] u[y].$$

Notice that for fixed a,b the inner sum is over the tensor product $\otimes \Gamma_{g_i}^{(a_i,b_i)}$. The largest eigenvalue of this matrix is $\prod_{i=1}^k \|\Gamma_{g_i}\|$, as $\|\Gamma_{g_i}^{(0,0)}\| = \|\Gamma_{g_i}^{(0,1)}\| = \|\Gamma_{g_i}^{(1,0)}\| = \|\Gamma_{g_i}^{(1,1)}\| = \|\Gamma_{g_i}\|$. It follows that,

$$\sum_{\substack{x,y\\x \in X_a, y \in X_b}} \left(\bigotimes_{i=1}^k \Gamma_{g_i}^{(a_i,b_i)} \right) [x,y] \cdot u[x] u[y] \le \prod_{i=1}^k \|\Gamma_{g_i}\| \cdot \|u_a\| \|u_b\|.$$

By the nonnegativity of Γ_f ,

$$u^* \Gamma_h u \leq \prod_{i=1}^k \|\Gamma_{g_i}\| \cdot \sum_{a,b} \Gamma_f[a,b] \cdot \|u_a\| \|u_b\|$$

$$\leq \prod_{i=1}^k \|\Gamma_{g_i}\| \cdot \|\Gamma_f\| \cdot \|u\|^2$$

$$= \|\Gamma_f\| \cdot \prod_{i=1}^k \|\Gamma_{g_i}\|.$$

We now turn to the lower bound. We wish to show that

$$\delta_h[x] = \delta_f[\tilde{x}] \cdot (\bigotimes_{i=1}^k \delta_{a_i}^{\uparrow \tilde{x}_i})[x]$$

is an eigenvector of Γ_h with eigenvalue $\|\Gamma_f\| \cdot \prod_i \|\Gamma_{g_i}\|$. As Γ_{g_i} is bipartite, notice that $\Gamma_{g_i} \delta_{g_i}^{\uparrow b} = \|\Gamma_{g_i}\| \delta_{g_i}^{\uparrow 1-b}$, for $b \in \{0,1\}$. As δ_{g_i} is a unit vector it follows that $\|\delta_{g_i}^{\uparrow 0}\|^2 = \|\delta_{g_i}^{\uparrow 1}\|^2 = 1/2$. Thus $\|\otimes \delta_{g_i}^{\uparrow a_i}\|^2 = \prod \|\delta_{g_i}^{\uparrow a_i}\| = 1/2^k$, for any $a \in \{0,1\}^k$. Hence also $\|\delta_h\|^2 = 1/2^k$.

Consider the sum

$$\delta_h^* \Gamma_h \delta_h = \sum_{a,b} (\delta_f[a] \cdot \otimes \delta_{g_i}^{\uparrow a_i})^* \left(\Gamma_f[a,b] \cdot \bigotimes \Gamma_{g_i}^{(a_i,b_i)} \right) (\delta_f[b] \cdot \otimes \delta_{g_i}^{\uparrow b_i}). \tag{3}$$

Notice that for fixed $a, b \in \{0, 1\}^k$

$$(\otimes \delta_{g_i}^{\restriction a_i})^* \bigotimes \Gamma_{g_i}^{(a_i,b_i)} (\otimes \delta_{g_i}^{\restriction b_i}) = \prod_{i=1}^k \|\Gamma_{g_i}\| \cdot \|\otimes \delta_{g_i}^{\restriction a_i}\|^2 = \frac{1}{2^k} \prod_{i=1}^k \|\Gamma_{g_i}\|. \tag{4}$$

To see this, consider the two cases $a_i = b_i$ and $a_i = 1 - b_i$:

- $\bullet \ \ \text{if} \ a_i=b_i \ \text{then} \ \delta_{q_i}^{\lceil a_i}=\delta_{q_i}^{\lceil b_i} \ \text{and} \ \Gamma_{g_i}^{(a_i,b_i)}=\|\Gamma_{g_i}\|I, \ \text{thus} \ (\delta_{g_i}^{\lceil a_i})^*\Gamma_{g_i}^{(a_i,b_i)}\delta_{g_i}^{\lceil b_i}=\|\Gamma_{g_i}\|\|\delta_{g_i}^{\lceil a_i}\|^2,$
- if $a_i = 1 b_i$ then $\Gamma_{g_i}^{(a_i,b_i)}$ sends $\delta_{g_i}^{\lceil b_i}$ to $\|\Gamma_{g_i}\|\delta_{g_i}^{\lceil a_i}$ and so $(\delta_{g_i}^{\lceil a_i})^*\Gamma_{g_i}^{(a_i,b_i)}\delta_{g_i}^{\lceil b_i} = \|\Gamma_{g_i}\|\|\delta_{g_i}^{\lceil a_i}\|^2$.

Substituting expression (4) into the sum (3) we have

$$\delta_{h}^{*}\Gamma_{h}\delta_{h} = \frac{1}{2^{k}} \prod_{i=1}^{k} \|\Gamma_{g_{i}}\| \sum_{a,b} \delta_{f}[a]^{*}\Gamma_{f}[a,b]\delta_{f}[b]$$

$$= \frac{1}{2^{k}} \prod_{i=1}^{k} \|\Gamma_{g_{i}}\| \cdot \|\Gamma_{f}\| \cdot \|\delta_{f}\|^{2}$$

$$= \|\Gamma_{f}\| \cdot \prod_{i=1}^{k} \|\Gamma_{g_{i}}\| \cdot \|\delta_{h}\|^{2}.$$

3 Composition lower bound

With Lemma 6 in hand, it is a relatively easy matter to show a lower bound on the adversary value of the composed function h.

Lemma 7 $ADV_{\alpha}(h) \geq ADV_{\beta}(f)$.

Proof. Due to the maximization over all matrices Γ , the spectral bound of the composite function h is at least $\mathrm{ADV}_{\alpha}(h) \geq \min_{\ell=1}^n (\alpha_{\ell} \|\Gamma_h\|/\|\Gamma_h \circ D_{\ell}\|)$, where Γ_h is defined as in Lemma 6. We compute $\|\Gamma_h \circ D_{\ell}\|$ for $\ell=1,\ldots,n$. Let the ℓ -th input bit be the q-th bit in the p-th block. Recall that

$$\Gamma_h[x,y] = \Gamma_f[\tilde{x},\tilde{y}] \cdot \prod_{i=1}^k \Gamma_{g_i}^{(\tilde{x}_i,\tilde{y}_i)}[x^i,y^i].$$

We prove that

$$(\Gamma_h \circ D_\ell)[x,y] = (\Gamma_f \circ D_p)[\tilde{x},\tilde{y}] \cdot (\Gamma_{g_p} \circ D_q)^{(\tilde{x}_p,\tilde{y}_p)}[x^p,y^p] \cdot \prod_{i \neq p} \Gamma_{g_i}^{(\tilde{x}_i,\tilde{y}_i)}[x^i,y^i].$$

If $x_{\ell} \neq y_{\ell}$ and $\tilde{x}_p \neq \tilde{y}_p$ then both sides are equal because all multiplications by D_p, D_q, D_{ℓ} are multiplications by 1. If this is not the case—that is, if $x_{\ell} = y_{\ell}$ or $\tilde{x}_p = \tilde{y}_p$ —then both sides are zero. We see this by means of two cases:

- 1. $x_{\ell} = y_{\ell}$: In this case the left hand side is zero due to $(\Gamma_h \circ D_{\ell})[x, y] = 0$. The right hand side is also zero because
 - (a) if $\tilde{x}_p = \tilde{y}_p$ then the right hand side is zero as $(\Gamma_f \circ D_p)[\tilde{x}, \tilde{y}] = 0$.
 - (b) else if $\tilde{x}_p \neq \tilde{y}_p$ then the right hand side is zero as $(\Gamma_{g_p} \circ D_q)[x^p, y^p] = 0$.

2. $x_{\ell} \neq y_{\ell}$, $\tilde{x}_p = \tilde{y}_p$: The left side is zero because $\Gamma_{g_p}^{(\tilde{x}_p,\tilde{y}_p)}[x^p,y^p] = \|\Gamma_{g_p}\|I[x^p,y^p] = 0$ since $x^p \neq y^p$. The right side is also zero due to $(\Gamma_f \circ D_p)[\tilde{x},\tilde{y}] = 0$.

Since $\Gamma_h \circ D_\ell$ has the same structure as Γ_h , by Lemma 6, $\|\Gamma_h \circ D_\ell\| = \|\Gamma_f \circ D_p\| \cdot \|\Gamma_{g_p} \circ D_q\| \cdot \|\Gamma_{i\neq p}\|\Gamma_{g_i}\|$. By dividing the two spectral norms,

$$\frac{\|\Gamma_h\|}{\|\Gamma_h \circ D_\ell\|} = \frac{\|\Gamma_f\|}{\|\Gamma_f \circ D_p\|} \cdot \frac{\|\Gamma_{g_p}\|}{\|\Gamma_{g_p} \circ D_q\|}.$$
 (5)

Since the spectral adversary maximizes over all adversary matrices, we conclude that

$$ADV_{\alpha}(h) \geq \min_{\ell=1}^{n} \frac{\|\Gamma_{h}\|}{\|\Gamma_{h} \circ D_{\ell}\|} \cdot \alpha_{\ell}$$

$$= \min_{i=1}^{k} \frac{\|\Gamma_{f}\|}{\|\Gamma_{f} \circ D_{i}\|} \cdot \min_{j=1}^{n_{i}} \frac{\|\Gamma_{g_{i}}\|}{\|\Gamma_{g_{i}} \circ D_{j}\|} \cdot \alpha_{j}^{i}$$

$$= \min_{i=1}^{k} \frac{\|\Gamma_{f}\|}{\|\Gamma_{f} \circ D_{i}\|} \cdot ADV_{\alpha^{i}}(g_{i})$$

$$= \min_{i=1}^{k} \frac{\|\Gamma_{f}\|}{\|\Gamma_{f} \circ D_{i}\|} \cdot \beta_{i}$$

$$= ADV_{\beta}(f),$$

which we had to prove.

4 Composition upper bound

In this section we prove the upper bound $ADV_{\alpha}(h) \leq ADV_{\beta}(f)$. We apply the duality theory of semidefinite programming to obtain an equivalent expression for ADV_{α} in terms of a minimization problem. We then upper bound $ADV_{\alpha}(h)$ by showing how to compose solutions to the minimization problems.

Definition 2 Let $f: S \to \{0,1\}$ be a partial boolean function, where $S \subseteq \{0,1\}^n$, and let $\alpha \in \mathbb{R}^n_+$. The minimax bound of f with costs α is

$$\mathrm{MM}_{\alpha}(f) = \min_{p} \max_{\substack{x,y \\ f(x) \neq f(y)}} \frac{1}{\sum_{i:x_i \neq y_i} \sqrt{p_x(i)p_y(i)}/\alpha_i},$$

where $p: S \times [n] \to [0,1]$ ranges over all sets of |S| probability distributions over input bits, that is, $p_x(i) \ge 0$ and $\sum_i p_x(i) = 1$ for every $x \in S$.

This definition is a natural generalization of the minimax bound introduced in [LM04, ŠS06]. As [ŠS06] show that the minimax bound is equal to the spectral norm formulation of the adversary method, one can similarly show that the versions of these methods with costs are equal.

Theorem 8 (Duality of adversary bounds) For every $f: \{0,1\}^n \to \{0,1\}$ and $\alpha \in \mathbb{R}^n_+$,

$$ADV_{\alpha}(f) = MM_{\alpha}(f).$$

Sketch of proof. We start with the minimax bound with costs, substitute $q_x(i)p_x(i)/\alpha_i$, and rewrite the condition $\sum_i p_x(i) = 1$ into $\sum_i \alpha_i q_x(i) = 1$. Using similar arguments as in [ŠS06], we rewrite the bound as a semidefinite program, compute its dual, and after a few simplifications, get the spectral bound with costs.

Lemma 9 $ADV_{\alpha}(h) \leq ADV_{\beta}(f)$.

Proof. Let p^f and p^{g_i} for $i=1,\ldots,k$ be optimal sets of probability distributions achieving the minimax bounds. Thus using Theorem 8 we have

$$ADV_{\beta}(f) = \max_{\substack{x,y \\ f(x) \neq f(y)}} \frac{1}{\sum_{i:x_{i} \neq y_{i}} \sqrt{p_{x}^{f}(i)p_{y}^{f}(i)}/\beta_{i}},$$

$$ADV_{\alpha^{i}}(g_{i}) = \max_{\substack{x,y \\ g_{i}(x) \neq g_{i}(y)}} \frac{1}{\sum_{j:x_{j} \neq y_{j}} \sqrt{p_{x}^{g_{i}}(j)p_{y}^{g_{i}}(j)}/\alpha_{j}^{i}}.$$

Define the set of probability distributions p^h as $p_x^h(\ell) = p_{\tilde{x}}^f(i)p_{x^i}^{g_i}(j)$, where the ℓ -th input bit is the j-th bit in the i-th block. This construction was first used by Laplante, Lee, and Szegedy [LLS06]. We claim that p^h witnesses that $ADV_{\alpha}(h) \leq ADV_{\beta}(f)$:

$$\begin{aligned} \operatorname{ADV}_{\alpha}(h) &\leq \max_{\substack{x,y \\ h(x) \neq h(y)}} \frac{1}{\sum_{\ell: x_{\ell} \neq y_{\ell}} \sqrt{p_{x}^{h}(\ell) p_{y}^{h}(\ell)} / \alpha_{\ell}} \\ &= 1 / \min_{\substack{x,y \\ h(x) \neq h(y)}} \sum_{\ell: x_{\ell} \neq y_{\ell}} \sqrt{p_{x}^{f}(i) p_{y}^{f}(i)} \sqrt{p_{x^{i}}^{g_{i}}(j) p_{y^{i}}^{g_{i}}(j)} / \alpha_{j}^{i} \\ &= 1 / \min_{\substack{x,y \\ f(x) \neq f(y)}} \sum_{i: x_{\ell} \neq y_{\ell}} \sqrt{p_{x}^{f}(i) p_{y}^{f}(i)} \min_{\substack{x^{i}, y^{i} \\ g_{i}(x^{i}) = \bar{x}_{i}}} \sum_{j: x_{j}^{i} \neq y_{j}^{i}} \sqrt{p_{x^{i}}^{g_{i}}(j) p_{y^{i}}^{g_{i}}(j)} / \alpha_{j}^{i} \\ &\leq 1 / \min_{\substack{\bar{x}, \bar{y} \\ f(\bar{x}) \neq f(\bar{y})}} \sum_{i: \bar{x}_{i} \neq \bar{y}_{i}} \sqrt{p_{\bar{x}}^{f}(i) p_{\bar{y}}^{f}(i)} \min_{\substack{x^{i}, y^{i} \\ g_{i}(x^{i}) \neq g_{i}(y^{i})}} \sum_{j: x_{j}^{i} \neq y_{j}^{i}} \sqrt{p_{x^{i}}^{g_{i}}(j) p_{y^{i}}^{g_{i}}(j)} / \alpha_{j}^{i} \\ &= 1 / \min_{\substack{\bar{x}, \bar{y} \\ f(\bar{x}) \neq f(\bar{y})}} \sum_{i: \bar{x}_{i} \neq \bar{y}_{i}} \sqrt{p_{\bar{x}}^{f}(i) p_{\bar{y}}^{f}(i)} / \operatorname{ADV}_{\alpha^{i}}(g_{i}) \\ &= 1 / \min_{\substack{\bar{x}, \bar{y} \\ f(\bar{x}) \neq f(\bar{y})}} \sum_{i: \bar{x}_{i} \neq \bar{y}_{i}} \sqrt{p_{\bar{x}}^{f}(i) p_{\bar{y}}^{f}(i)} / \beta_{i} \\ &= \operatorname{ADV}_{\beta}(f), \end{aligned}$$

where the second inequality follows from that fact that we have removed $i: \tilde{x}_i = \tilde{y}_i$ from the sum and the last equality follows from Theorem 8.

Laplante, Lee, and Szegedy [LLS06] proved a similar bound in a stronger setting where the sub-functions g_i can act on the same input bits. They did not allow costs of input bits. This setting is, however, not applicable to us, because we cannot prove a matching lower bound for $ADV_{\alpha}(h)$.

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